TWO-DIMENSIONAL GRADIENT RICCI SOLITONS REVISITED

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Abstract. In this note, we complete the classification of the geometry of non-compact two-dimensional gradient Ricci solitons. As a consequence, we obtain two corollaries: First, a complete two-dimensional gradient Ricci soliton has bounded curvature. Second, we give examples of complete two-dimensional expanding Ricci solitons with negative curvature that are topologically disks and are not hyperbolic space.

1. Introduction

Recall, a triple (M, g, X) consisting of a smooth manifold M, a smooth Riemannian metric g on M and a smooth vector field $X \in \Gamma(TM)$ is said to be a *Ricci soliton with expansion constant* λ, if it satisfies

$$-2\operatorname{Ric}(g) = L_X g - 2\lambda g \tag{1.1}$$

where $L_X g$ denotes the Lie derivative of g with respect to X. For such a triple the Ricci flow equation

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t = -2\operatorname{Ric}(g_t)$$

with initial condition $g_0 = g$ has, in the t-interval on which $1-2\lambda t > 0$, the solution

$$g_t = (1-2\lambda t)\phi_t^* g,$$

where $\phi_t : M \to M$ is the (possibly locally defined) time t flow of X. When $\lambda = 0$, the soliton is said to be steady, when $\lambda > 0$, the soliton is said to be shrinking, and when $\lambda < 0$, the soliton is said to be expanding.

Let ∇ denote the Levi-Civita connection of g. If $X = \nabla f$ for some smooth real-valued function f on M, then we say (M, g, f) is a *gradient Ricci soliton* and f is its *soliton potential*.

It has been known for some time that closed two-dimensional Ricci solitons must have constant curvature – see [9, Theorem 10.1] and [10] – also [4] and [13]. It was also well-known that any two-dimensional gradient Ricci soliton must admit a non-trivial Killing vector field – see, for instance, the editor's footnote of [3, p. 241-242]. However, the possible geometries in the open case were not completely determined.

The most complete analysis in this direction was in [12, Appendix A and B] where all rotationally symmetric two-dimensional self-similar solutions to the logarithmic fast diffusion equation were explicitly computed. As the logarithmic fast diffusion

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equation is the evolution of the conformal factor – with respect to a fixed flat background – of a metric evolving under the Ricci flow, the analysis of [12] gives many examples of two-dimensional Ricci solitons. More in the spirit of the present note is [11], where the Killing vector field is used to show that all positively curved two-dimensional expanding Ricci solitons are the rotationally symmetric examples of [8]. A more complete overview of known results may be found in [6, Chap. 1].

Motivated by an observation of [2] connecting two-dimensional steady gradient Ricci solitons with minimal surface metrics, we reduce the classification of two-dimensional gradient Ricci solitons to the analysis of a simple autonomous first order ODE. The analysis of this ODE gives:

Theorem 1. *The following twelve one-parameter families of rotationally symmetric metrics are gradient Ricci soliton metrics on either the disks* $\mathbb{D} = \{x : |x| < 1\} \subset \mathbb{R}^2$ *or* $\mathbb{R}^2$ *or the annuli* $\mathbb{D}_* = \mathbb{D}\setminus\{0\}$ *or* $\mathbb{R}^2_* = \mathbb{R}^2\setminus\{0\}$*:*

(1) *Steady solitons:*
- (a) $(\mathbb{R}^2, g_1(\nu))$ – *The cigar solitons of Hamilton* [9] – *i.e. complete positively curved steady solitons – the parameter* ν *corresponds to scaling of the metric;*
- (b) $(\mathbb{R}^2, g_2(\nu))$ – *Incomplete steady solitons with unbounded negative curvature – called the exploding solitons in* [6] – *the parameter* ν *corresponds to scaling of the metric;*
- (c) $(\mathbb{R}^2_*, g_3(\nu))$ – *Incomplete negatively curved steady solitons asymptotic to a cylinder of radius* $\nu \geq 0$ *at* 0 *and to* $g_2(1)$ *at* ∞*.*

(2) *Shrinking solitons:*
- (a) $(\mathbb{D}, g_4^{\pm}(\nu))$ – *Two families of incomplete positively curved shrinking solitons which extend smoothly to* $\bar{\mathbb{D}}$ *so that* $\partial\bar{\mathbb{D}}$ *is totally geodesic and has length* 2π – *the parameter* $\nu = \mathrm{dist}(0, \partial\mathbb{D})$ *satisfies* $\nu \in (1, \infty)$ *for* g_4^+ *and* $\nu \in (\frac{\pi}{2}, \infty)$ *for* g_4^-*;*
- (b) $(\mathbb{R}^2, g_5(\nu))$ – *Incomplete negatively curved shrinking solitons asymptotic to* $g_2(\nu)$*;*

(3) *Expanding solitons:*
- (a) $(\mathbb{R}^2, g_6(\nu))$ – *Complete positively curved expanding solitons asymptotic to the end of a flat cone of angle* $\nu \in (0, 2\pi)$ – *see* [8]*;*
- (b) $(\mathbb{R}^2, g_7(\nu))$ – *Complete negatively curved expanding solitons asymptotic to to the end of a flat cone of angle* $\nu \in (2\pi, \infty)$*;*
- (c) $(\mathbb{R}^2_*, g_8(\nu))$ – *Complete negatively curved expanding solitons asymptotic to a hyperbolic cusp at* 0 *and to the end of a flat cone of angle* $\nu \in (0, \infty)$ *at* ∞*;*
- (d) $(\mathbb{D}_*, g_9(\nu))$ – *Incomplete negatively curved expanding solitons asymptotic to to the end of a flat cone of angle* $\nu \in (0, \infty)$ *at* 0 *and extending smoothly to* $\bar{\mathbb{D}}_*$ *so that* $\partial\bar{\mathbb{D}}$ *is totally geodesic and of length* 2π*;*
- (e) $(\mathbb{R}^2, g_{10}(\nu))$ – *Incomplete negatively curved expanding solitons asymptotic to* $g_2(\nu)$*;*
- (f) $(\mathbb{R}^2_*, g_{11}(\nu))$ – *Incomplete negatively curved expanding solitons asymptotic to a hyperbolic cusp at* 0 *and to* $g_2(\nu)$ *at* ∞*.*
- (g) $(\mathbb{D}_*, g_{12}(\nu))$ – *Incomplete negatively curved expanding solitons asymptotic to* $g_2(\nu)$ *at* 0 *and extending smoothly to* $\bar{\mathbb{D}}_*$ *so* $\partial\bar{\mathbb{D}}$ *is totally geodesic and of length* 2π*;*

Remark 1. The metrics $g_4(\nu), g_9(\nu)$ and $g_{12}(\nu)$ may be "doubled" to give $C^{2,1}$ metrics on a doubled surface. The doubled metrics are not themselves Ricci solitons.

Remark 2. All these metrics seem to have been written down in [12], however their geometric properties are not fully discussed there.

A further consequence of the analysis of the ODE is that the above list contains all possible models of non-constant curvature two-dimensional gradient Ricci solitons.

Theorem 2. *Let (M, g, f) be a smooth gradient Ricci soliton with M a connected orientable two-manifold. If $M_* = \{p \in M : \nabla_g K_g \neq 0\}$ is non-empty and $\pi_1(M_*)$ is cyclic, then there is a $0 < R \leq \infty$, a metric $\bar{g}$ on $\mathbb{D}_*(R) = \{x : 0 < |x| < R\} \subset \mathbb{R}^2$ and an isometric immersion $i : M_* \to \mathbb{D}_*(R)$ so that:*

1. *$K_{\bar{g}} \neq 0$;*
2. *$(\mathbb{D}_*(R), \bar{g})$ is maximal in the sense that if $(\hat{M}, \hat{g}, \bar{f})$ is a two-dimensional connected gradient Ricci soliton and $\hat{i} : (\mathbb{D}_*(R), \bar{g}) \to (\hat{M}, \hat{g})$ is a smooth isometric embedding, then $\hat{M} \backslash \hat{i}(\mathbb{D}_*(R)) = \{p\}$ and $\nabla_{\hat{g}} K_{\hat{g}}(p) = 0$;*
3. *$\bar{g}$ is rotationally symmetric, that is in polar coordinates (r, θ)*
$$\bar{g} = \mathrm{d}r^2 + b^2(r)\mathrm{d}\theta^2$$
for some positive functions $b \in C^\infty((0, R))$;
4. *There are $\alpha, \beta > 0$ so that if $\bar{g}_\alpha = \mathrm{d}r^2 + \alpha^2 b^2(r)\mathrm{d}\theta^2$, then $(\mathbb{D}_*(R), \beta^2 \bar{g}_\alpha)$ may be isometrically embedded in one of the models given in Theorem 1.*

Remark 3. If $g = \mathrm{d}r^2 + b^2(r)\mathrm{d}\theta^2$ is a metric on $\mathbb{D}_*(R)$, then for any $\alpha > 0$ the metric $g_\alpha = \mathrm{d}r^2 + \alpha^2 b^2(r)\mathrm{d}\theta^2$ on $\mathbb{D}_*(R)$ is locally isometric to g. In particular, g is a gradient Ricci soliton metric if and only if g_α is. Likewise scaling a Ricci soliton metric gives a new Ricci soliton metric with scaled expansion constant.

Remark 4. Neither the orientability condition on M nor the condition on $\pi_1(M_*)$ can be removed because several locally isometric but not globally isometric subsets of a fixed model soliton may be glued together to yield counter-examples.

Theorems 1 and 2 resolve two questions regarding the structure of non-compact two-dimensional solitons (see [6, p. 61]). Specifically,

Corollary 1. *A complete two-dimensional gradient Ricci soliton has bounded curvature.*

Corollary 2. *There exist complete two-dimensional expanding Ricci solitons with negative curvature that are topologically disks and are not hyperbolic space – namely the metrics g_7 and the universal covers of the metrics g_8.*

Remark 5. The metrics g_7 are parabolic and appear in [12] – though this does not seem to be widely known. The universal covers of the metrics g_8 are non-parabolic.

Finally, in Section 4 we give a simple variational characterization of two-dimensional gradient Ricci solitons and observe a connection with [2].

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2. Properties of two-dimensional Ricci solitons

We will henceforth assume M to be a connected two-manifold. Then for a gradient Ricci soliton equation (1.1) becomes

$$\nabla^2 f + Kg - \lambda g = 0$$

which is equivalent to

$$\Delta f = 2(\lambda - K) \quad \text{and} \quad \mathring{\nabla}^2 f = 0.$$

Here K denotes the Gauss curvature, Δ the Laplacian, and $\mathring{\nabla}^2$ the trace-free Hessian of g. Following, [2, Appendix A], covariant differentiation of $\mathring{\nabla}^2 f = 0$ together with straightforward manipulations gives

$$0 = K\mathrm{d}f + \frac{1}{2}\mathrm{d}\Delta f = K\mathrm{d}f - \mathrm{d}K.$$

At a point $p \in M$ where $K \neq 0$ we thus have

$$\mathrm{d}f = \mathrm{d}\log|K|$$

which yields

$$\nabla^2 \log|K| = \nabla^2 f = (\lambda - K)g.$$

From this we see that

$$\Delta \log|K| = 2(\lambda - K) \quad \text{and} \quad \mathring{\nabla}^2 \log|K| = 0. \tag{2.1}$$

We have shown:

Proposition 3. *Let (M, g) be a Riemannian two-manifold with non-zero Gauss curvature. Then g is a gradient Ricci soliton metric with expansion constant λ if and only if*

$$\mathring{\nabla}^2 \log|K| = 0 \quad \textit{and} \quad \Delta \log|K| = 2(\lambda - K).$$

Moreover, if (M, g, f) is a gradient Ricci soliton, then we may take $f = \log|K|$.

Remark 6. Recall (c.f. [6, p. 11]) that if $(M, g, \nabla f)$ is a gradient Ricci soliton and M is an oriented surface, then $J(\nabla f)$ is a Killing vector field for g. Here, as usual, J denotes the integrable almost complex structure induced by g and the orientation, i.e. J rotates $v \in TM$ by $\pi/2$ in positive direction.

Proposition 3 and Remark 6 reduce the classification of gradient Ricci solitons in dimension two to the study of a simple ODE. Indeed, assume (M, g) is a gradient Ricci soliton and g does not have constant curvature. We first observe that M has non-vanishing curvature and has non-constant curvature almost everywhere.

Proposition 4. *If (M, g, f) is a connected two-dimensional gradient Ricci soliton and $M_* = \{p \in M : \nabla K(p) \neq 0\} \neq \emptyset$, then $K \neq 0$ on all of M and $M \backslash M_*$ consists of isolated points.*

Proof. Set $M_{**} = \{p \in M : K(p) \neq 0, \nabla K(p) \neq 0\}$ and decompose $B = M \backslash M_{**}$ into $B_1 = \{p \in M : K(p) = 0, \nabla K(p) \neq 0\}$, $B_2 = \{p \in M : K(p) = 0, \nabla K(p) = 0\}$ and $B_3 = \{p \in M : K(p) \neq 0, \nabla K(p) = 0\}$. As M has non-constant curvature, M_{**} is non-empty. We claim it is dense in M. This follows from standard unique continuation results for uniformly elliptic equations [1] and the fact that K satisfies

$$\Delta K - \nabla f \cdot \nabla K + 2K^2 + \lambda K = 0$$

on M – see [6, Lemma 1.11]. Hence, $B_1 = \emptyset$. Indeed, by Proposition 3, $\nabla f - \nabla \log|K_g|$ is covariant constant on M_{**} and so is bounded on all of M.

Fix $p \in M$ and let (r, θ) be normal coordinates about p. In these coordinates,

$$\Delta u = \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u + \alpha_1 \partial_r u + \frac{1}{r}\alpha_2 \partial_\theta u$$

where α_i are smooth functions – depending on u and g – defined near p. We next show that $B_2 = \emptyset$. Indeed, by [6, Lemma 1.11], the unique continuation result [1] and the expansion of the Laplacian in normal coordinates near any $p \in B_2$ we have

$$K = \alpha_1 r^n \cos n\theta + \alpha_2^2 r^n \beta \sin n\theta + o(r^n)$$

for some $n \geq 2$ and $\alpha_1^2 + \alpha_2^2 \neq 0$. However, if this is true then there would exist a point in B_1 and so $B_2 = \emptyset$.

Finally, we conclude that B_3 consists of isolated points. Indeed, $\nabla \log|K_g|$ is a conformal Killing vector field on M and so is (locally) the real part of a non-trivial holomorphic vector field. In particular, it can only vanish at isolated points. □

By Proposition 3, in a neighborhood of every point $p \in M_*$ there exists a non-trivial Killing vector field for g. It follows – see [6, Lemma 1.18] – that there exists a local coordinate system (r, θ) defined on some neighborhood U_p of p, such that

$$g = \mathrm{d}r^2 + b(r)^2 \mathrm{d}\theta^2$$

for some smooth positive real-valued function b defined in some neighborhood of $r(p)$. On U_p we thus have

$$K(r) = -\frac{b''(r)}{b(r)}.$$

Since $K(p) \neq 0$ we can assume (after possibly replacing p with a nearby point), that $b'(r)$ is non-vanishing in some simply connected neighborhood V_p of p. Setting $t = \frac{1}{4}b(r)^2$ it follows that $(t, \theta) : V_p \to \mathbb{R}^2$ is a local coordinate system so that

$$g = \frac{a(t)^2}{t}\mathrm{d}t^2 + 4t\mathrm{d}\theta^2 \tag{2.2}$$

for some positive function a defined in some neighborhood of $t(p)$. For $t > 0$ in the domain of definition of a, this metric is smooth and has Gaussian curvature

$$K(t) = \frac{1}{2}\frac{a'(t)}{a(t)^3}. \tag{2.3}$$

For later reference we remark that the geodesic curvature of the curves $\{t = t_0\}$ is

$$\kappa = \frac{1}{2\sqrt{t_0 a(t_0)}} \tag{2.4}$$

which follows easily from the first variation formula.

As $J(\nabla \log|K|)$ is a Killing vector field, we may take

$$\frac{a(t)}{2t}\frac{t}{a(t)^2}\left(\log\left|\frac{1}{2}\frac{a'(t)}{a(t)^3}\right|\right)' = 2\mu \tag{2.5}$$

for some constant μ which is non-vanishing since $\mathrm{d}K(p) \neq 0$. On the other hand, from Proposition 3 we obtain

$$\frac{1}{2a(t)}\left(\frac{2t}{a(t)}\left(\log\left|\frac{1}{2}\frac{a'(t)}{a(t)^3}\right|\right)'\right)' = -\frac{a(t)'}{a(t)^3} + 2\lambda. \tag{2.6}$$

Combining (2.5) and (2.6) gives the autonomous first-order ODE:

$$a'(t) = 4\mu a(t)^2 \left(\frac{\lambda}{2\mu} a(t) - 1\right). \tag{2.7}$$

Conversely we have:

Lemma 5. *If $a > 0$ is defined on $(0, T)$ with $0 < T \leq \infty$ and solves (2.7) for some $\lambda, \mu \neq 0$, then every solution b to*

$$a\left(\frac{b(r)^2}{4}\right) b'(r) = 1, \tag{2.8}$$

defined on $(0, R)$ gives rise to a smooth rotationally symmetric gradient Ricci soliton $g = \mathrm{d}r^2 + b(r)^2 \mathrm{d}\theta^2$ on $\mathbb{D}_*(R)$. Furthermore, there exists a solution b to (2.8) so that the associated metric extends smoothly to $\mathbb{D}(R)$ if and only if $\lim_{t\to 0} a(t) = 1$. In this case $K(0) = \lambda - 2\mu$ and $\nabla K(0) = 0$.

Proof. The first statement is an immediate consequence of the calculations above. To prove the second statement we first recall the standard result – see [6, Lemma A.2] – that a metric of the form $g = \mathrm{d}r^2 + b(r)^2 \mathrm{d}\theta^2$ for some smooth non-vanishing function b on $(0, R)$ extends smoothly to $\mathbb{D}(R)$ if and only if

$$\lim_{r\to 0} b'(r) = \pm 1, \quad \lim_{r\to 0} \frac{\mathrm{d}^{2k} b(r)}{\mathrm{d}r^{2k}} = 0 \ \ \forall k \in \mathbb{N}_0. \tag{2.9}$$

Suppose the metric associated to a solution of (2.8) extends smoothly to the topological disk, then it follows with the continuity of the function a and (2.8, 2.9) that $\lim_{t\to 0} a(t) = 1$.

Conversely, suppose the function a satisfies $\lim_{t\to 0} a(t) = 1$ so that a smoothly extends to a solution of (2.7) on the interval $(-\varepsilon, T)$ for some $\varepsilon \in (0, \infty]$. Choosing the initial condition $b(0) = 0$ we obtain a solution to (2.8) defined in some neighborhood of 0 which satisfies $b'(0) = 1$. Thus, there exists an interval $(-R, R)$ about $t = 0$ on which b is an odd function of r.

We obtain from (2.3) and (2.7)

$$K(t) = \lambda - \frac{2\mu}{a(t)} \tag{2.10}$$

which, assuming $\lim_{t\to 0} a(t) = 1$, converges to $\lambda - 2\mu$ as $t \to 0$. Because of the rotational symmetry of the metric g, we have $\nabla K(0) = 0$. □

More generally, we have the following result relating the qualitative behavior of solutions of (2.7) to the geometry of the metrics (2.2).

Lemma 6. *If $a > 0$ is defined on (T_0, T_1) with $0 \leq T_0 < T_1 \leq \infty$ and solves (2.7) for some $\lambda, \mu \neq 0$, then the following relationships hold between the behavior of a and the geometry of the metric g given by (2.2):*

1. *g has positive curvature if and only if a is strictly increasing and has negative curvature if and only if a is strictly decreasing;*
2. *g has bounded curvature if and only if a is strictly bounded away from 0;*
3. *g is complete if and only if*
   (a) *$0 = T_0 < T_1 < \infty$, $a(0) = 1$ and $\int_0^{T_1} \frac{a(t)}{t} dt = \infty$; or*
   (b) *$0 = T_0 < T_1 = \infty$, $a(0) = 1$ and a is strictly bounded away from 0; or*
   (c) *$0 < T_0 < T_1 = \infty$, $\int_{T_0}^{2T_0} \frac{a(t)}{t} dt = \infty$ and a is strictly bounded away from 0.*

(4) *if* $T_1 = \infty$ *and* $\lim_{t\to\infty} a(t) = a_\infty > 0$, *then* g *is asymptotic to the end of a flat cone of angle* $2\pi a_\infty^{-1}$;

Proof. Item (1) is an immediate consequence of (2.3). The converse direction of Item (2) follows from (2.10). On the other hand, if a is not strictly bounded away from the zero, then, as (2.7) is an autonomous first order ODE, $T_1 = \infty$ and $\lim_{t\to\infty} a(t) = 0$ so $K \to -\infty$. Item (3a) is an immediate consequence of the form of the metric and Lemma 5. Likewise, Items (3b) and Items (3c) will follow from the form of the metric and Lemma 5 provided we show that if $\lim_{t\to\infty} a(t) = 0$, then $\int_{T_0+1}^{\infty} \frac{a(t)}{t} dt < \infty$. However, from (2.7) it is clear that if $\lim_{t\to\infty} a(t) = 0$, then in all cases there is a constant $C_1 > 0$, depending on μ and λ, and a $T_* > T_0$, depending on a, so for $t > T_*$,

$$a'(t) \le -C_1 a(t)^3.$$

Standard ODE comparison then implies that for $t > T_*$, we have

$$a(t) \le \frac{1}{\sqrt{C_1 t - C_2}}$$

for some C_2 which proves the claim.

If $\lim_{t\to\infty} a(t) = a_\infty$, then $a(t) = a_\infty + O(e^{-t})$ and so $\int_{2T_0}^{\infty} \frac{a(t)}{t} = \infty$. Hence, by (2.8) b is definte on (R_0, ∞) and has the asymptotics

$$b(r) = a_\infty^{-1} r + O(1).$$

so the associated metric is asymptotic to the end of the claimed cone. □

3. The classification

We now analyze the solutions of (2.7) in order to prove Theorem 1. The constant solution $a \equiv 0$ separates the space of solutions of (2.7) into positive and negative solutions. For our purposes it sufficies to study positive solutions. Let $\mathcal{S}^+$ be the 3-dimensional space of triples (μ, γ, a) where $\mu, \gamma \in \mathbb{R}_*$ − γ possibly infinite − and a is a smooth positive real-valued function defined on some interval I which solves

$$a'(t) = 4\mu a(t)^2 \left(\frac{a(t)}{\gamma} - 1 \right). \tag{3.1}$$

There are three natural symmetries of $\mathcal{S}^+$. First, we let $\mathbb{R}^+$ act by

$$\alpha \cdot (\mu, \gamma, a) = \left(\frac{\mu}{\alpha^2}, \gamma, \alpha a \right)$$

The metrics g associated to a and g_α associated to αa are locally isometric, but not globally isometric. Second, we let $\mathbb{R}^+$ act by

$$(\mu, \gamma, a) \cdot \beta = \left(\frac{\mu}{\beta^4}, \gamma\beta^2, a \circ s_\beta \right)$$

where s_β denotes the map $t \mapsto t\beta^{-2}$. This corresponds to scaling the metric g associated to a by β^2. By Lemma 5, if $0 \in I$ and $a(0) = 1$, then this is also true after acting by β. Third, is the non-geometric symmetry where $\mathbb{R}$ acts by translation in t, i.e. $a(t) \mapsto a(t - \tau)$ for $\tau \in \mathbb{R}$. This action leaves μ, γ unchanged.

There are six orbits of the actions corresponding to $(\mu, \gamma) \in \{-1, 1\} \times \{-1, \infty, 1\}$ and within each orbit there are one-parameter families of geometrically distinct solitons. Due to the fact that certain solutions to (3.1) blow-up, some orbits contain more than one such family.

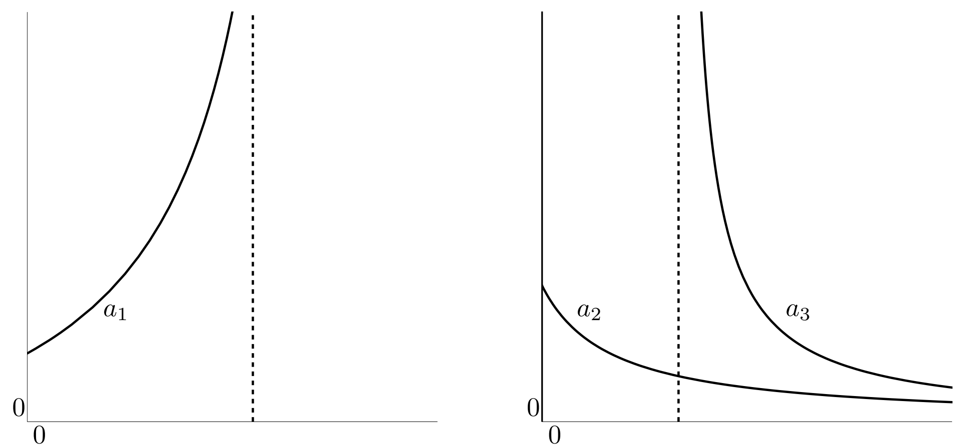

FIGURE 1. On the left solutions with $\gamma = \infty$ and $\mu < 0$ on the right solutions with $\gamma = \infty$ and $\mu > 0$.

3.1. **The case where $\gamma = \infty$.** In this case $\lambda = 0$ so we are considering metrics of steady solitons. Moreover, (2.7) simplifies to

$$a'(t) = -4\mu a(t)^2$$

whose solutions are given explicitly by

$$a(t) = \frac{1}{4\mu t + \varphi}$$

for some constant φ. Qualitatively, this 2-parameter family of solutions splits into three types, depicted in Figure 3.1.

3.1.1. $\mu = -\nu^2 < 0$. We may assume $\varphi > 0$, otherwise $a < 0$. It follows that $a' > 0$ is strictly increasing and hence the solutions blow up in finite time. Using the smooth extension condition of Lemma 5 we obtain the solutions

$$a_1(t) = \frac{1}{1 - 4\nu^2 t}$$

whose associated complete metrics $g_1(\nu)$ are Hamilton's steady *cigar solitons* [9] which are defined on the topological disk and given in polar coordinates by

$$g_1(\nu) = \mathrm{d}r^2 + \frac{1}{\nu^2}\tanh^2(\nu r)\mathrm{d}\theta^2.$$

By inspection these solutions are complete and have bounded positive curvature and are asymptotic to a cylinder of radius ν. Moreover, $g_1(\nu) = \nu^2 g_1(1)$.

3.1.2. $\mu = \nu^2 > 0$ *and* $\varphi > 0$. The solutions exist for all (positive) times, are bounded, and approach 0 as $t \to \infty$. It follows from Lemma 6 that the associated

metrics have unbounded negative Gauss curvature and are incomplete. Using the smooth extension condition of Lemma 5 we obtain the solutions

$$a_2(t) = \frac{1}{1+4\nu^2 t}$$

whose associated family of metrics – called *exploding solitons* in [6] – $g_2(\nu)$ are given in polar coordinates by

$$g_2(\nu) = \mathrm{d}r^2 + \frac{1}{\nu^2}\tan^2(\nu r)\mathrm{d}\theta^2.$$

Clearly, $g_2(\nu) = \nu^2 g_2(1)$. Furthermore, at $r = \frac{\pi}{2\nu}$ the metrics have the asymptotics

$$g_2(\nu) = \mathrm{d}r^2 + \left(\frac{1}{\nu^4\left(r - \frac{\pi}{2\nu}\right)^2} + O(1)\right)\mathrm{d}\theta^2.$$

3.1.3. $\mu > 0$ *and* $\varphi \leq 0$. The positive part of the solution exists on the time interval $(-\frac{\varphi}{4\mu}, \infty)$, and approaches ∞ as $t \to -\frac{\varphi}{4\mu}$ and 0 as $t \to \infty$. By Lemma 6, the associated metrics g_3 have unbounded negative Gauss curvature and are incomplete. Writing $\nu^2 = -\varphi$, and applying the β action with $\beta = \sqrt{\mu}$, gives

$$a_3(t) = \frac{1}{4t - \nu^2}.$$

In cylindrical coordinates (ρ, θ) this gives metrics $g_3(\nu)$ are given for $\nu > 0$ as

$$g_3(\nu) = \mathrm{d}\rho^2 + \frac{\nu^2}{\tanh^2(\nu\rho)}\mathrm{d}\theta^2$$

while for $\nu = 0$

$$g_3(0) = \mathrm{d}\rho^2 + \frac{1}{\rho^2}\mathrm{d}\theta^2.$$

Thus, the metrics are all asymptotically cylindrical with radius ν as $\rho \to -\infty$. At $\rho = 0$, the metrics are asymptotically

$$g_3(\nu) = \mathrm{d}\rho^2 + \left(\frac{1}{\rho^2} + O(1)\right)\mathrm{d}\theta^2$$

and so to leading order behave like $g_2(1)$ as desired.

3.2. **The case where $\infty > \gamma > 0$.** Besides the separatrix $a \equiv 0$, there is a positive separatrix $a \equiv \gamma$ and the orbits split into six families, depicted in Figure 3.2.

3.2.1. $\mu > 0$ *and* $a > \gamma > 0$. In this case the separatrix $a \equiv \gamma$ is unstable and so solutions starting above it are strictly increasing. If $1 > \gamma$, then there is a solution a_4^+ which satisfies $a_4^+(0) = 1$. An ODE comparison argument implies that the a_4^+ blow-up at some finite time $T_1(\mu, \gamma) > 0$ given by

$$4\mu T_1(\mu, \gamma) = -1 - \frac{1}{\gamma}\log(1-\gamma).$$

By applying an appropriate β action, we may take $T_1 = \frac{1}{4}$ and so $\mu = \mu(\gamma)$ and $\lambda = \lambda(\gamma)$. The solutions have the following asymptotics at $T_1 = \frac{1}{4}$:

$$a_4^+(t) = \frac{1}{\sqrt{\lambda(1-4t)}} + \frac{\gamma}{3} + o\left(1\right).$$

Clearly, a_4^+ is integrable and hence Lemma 5 and Lemma 6 imply that there are positively curved, incomplete metrics, $g_4^+(\gamma)$, on the disk $\mathbb{D}$.

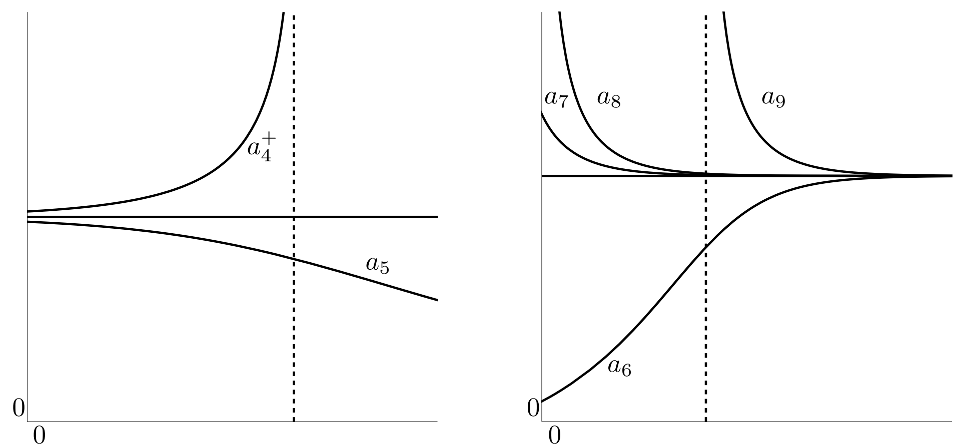

FIGURE 2. On the left solutions with $\infty > \gamma > 0$ and $\mu > 0$ on the right solutions with $\infty > \gamma > 0$ and $\mu < 0$.

One verifies that each $g_4^+(\gamma)$ extends smoothly to $\bar{\mathbb{D}}$ and, as $T_1 = \frac{1}{4}$ the length of $\partial\mathbb{D}$ is 2π. Moreover, it follows directly from (2.4) that $\partial\mathbb{D}$ is totally geodesic. Finally, $g_4^+(\gamma)$ converges to the flat metric on the disk $\mathbb{D}(1)$ as $\gamma \to 0$, while $g_4^+(\gamma)$ converges to the metric $g_1(1)$ as $\gamma \to 1$. Hence, $\operatorname{dist}(0, \partial\mathbb{D}) \to \infty$ as $\gamma \to 0$ while $\operatorname{dist}(0, \partial\mathbb{D}) \to 1$ as $\gamma \to 1$ and $\operatorname{dist}(0, \partial\mathbb{D})$ is a strictly decreasing function of γ. We take the parameter ν to be this distance.

3.2.2. $\mu > 0$ *and* $a < \gamma$. Again the separatrix $a \equiv \gamma$ is unstable and so solutions starting below it are decreasing and tend to 0 as $t \to \infty$. If $1 < \gamma$, then there is a solution a_5 which satisfies $a_5(0) = 1$ and so, by Lemma 5 and Lemma 6, there exist negatively curved, incomplete metrics $g_5(\gamma, \mu)$ with unbounded curvature on $\mathbb{R}^2$.

Furthermore, a_5 has the asymptotic form as $t \to \infty$

$$a_5(t) = \frac{1}{1+4\mu t} + \frac{1}{\gamma}\frac{\log(1+4\mu t)}{(1+4\mu t)^2} + o\left(\frac{\log(1+4\mu t)}{(1+4\mu t)^2}\right).$$

Hence, $a_5(t)$ is a asymptotic to $a_2(t)$ as $t \to \infty$ and so taking $\mu = \nu^2$, $g_5(\nu)$ is asymptotic to $g_2(\nu)$ as claimed.

3.2.3. $\mu < 0$ *and* $a < \gamma$. In this case the separatrix $a \equiv \gamma$ is stable and so solutions starting below it are increasing and $a(t) \to \gamma$ as $t \to \infty$. If $1 < \gamma$, then there is a solution a_6 which satisfies $a_6(0) = 1$ and so, by Lemma 5 and Lemma 6, there exists a family of positively curved, complete metrics $g_6(\nu)$ on $\mathbb{R}^2$ which are asymptotic to a cone of angle $\nu = 2\pi\gamma^{-1} \in (0, 2\pi)$.

3.2.4. $\mu < 0$ *and* $a > \gamma > 0$. In this case the separatrix $a \equiv \gamma$ is stable and so solutions starting above it are decreasing and $a(t) \to \gamma$ as $t \to \infty$. An ODE comparison argument implies that solutions blow-up at some finite initial time T_0.

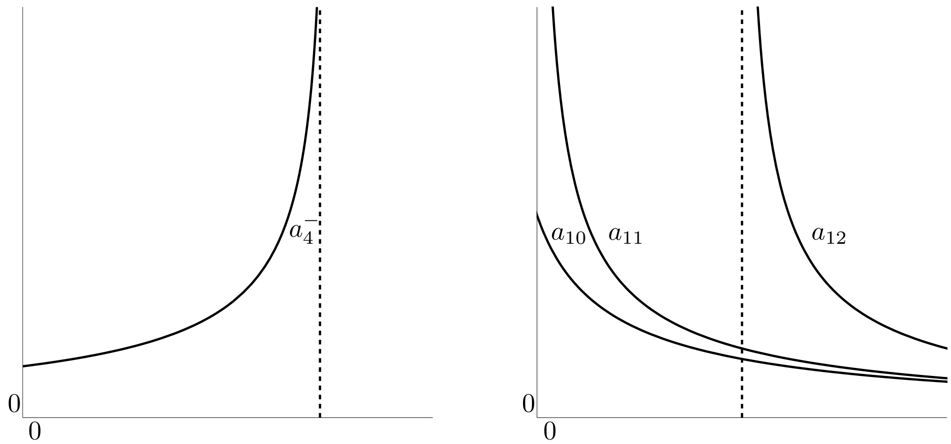

FIGURE 3. On the left solutions with $-\infty < \gamma < 0$ and $\mu > 0$ on the right solutions with $-\infty < \gamma < 0$ and $\mu < 0$.

The geometry of the associated metrics is quite different depending on whether $T_0 < 0$, $T_0 = 0$ or $T_0 > 0$.

If $T_0 < 0$ and $1 > \gamma$, then there is a solution a_7 which satisfies $a_7(0) = 1$ and so, by Lemma 5 and Lemma 6, there exists a family of negatively curved, complete metrics $g_7(\nu)$ on $\mathbb{R}^2$ which are asymptotic to a cone of angle $\nu = 2\pi\gamma^{-1} \in (2\pi, \infty)$.

If $T_0 = 0$, then any solution a_8 has the following asymptotics near $T_0 = 0$

$$a_8(t) = \frac{1}{\sqrt{-4\lambda t}} + o\left(\frac{1}{\sqrt{t}}\right)$$

and so $\frac{a_8(t)}{t}$ is not locally integrable. By the β symmetry we may take $\lambda = -1$. Hence, by Lemma 6, there is a family of negatively curved, complete metrics $g_8(\nu)$ on a cylinder which are asymptotic at one end to a cone with angle $\nu = 2\pi\gamma^{-1} \in (0, \infty)$. By (2.10), $K \to -1$ as $t \to 0$, while the length of the curve $\{t = t_0\}$ goes to 0 as $t_0 \to 0$ and hence $g_8(\nu)$ is asymptotic at the other end to a hyperbolic cusp.

If $T_0 > 0$, then any solution a_9 has the following asymptotics near $T_0 > 0$

$$a_9(t) = \frac{1}{\sqrt{-4\lambda(t - T_0)}} + o\left(\frac{1}{\sqrt{t - T_0}}\right)$$

and so $\frac{a_9(t)}{t}$ is locally integrable. Hence using Lemma 6, it follows that there is a family of negatively curved, incomplete metrics on $\mathbb{D}_*$ which are asymptotic to a cone with angle $\nu = 2\pi\gamma^{-1}$ at the puncture. As with the family $g_4(\nu)$, the metrics smoothly extend to $\bar{\mathbb{D}}_*$ and $\partial\bar{\mathbb{D}}^*$ is totally geodesic. By the β symmetry we may take the length of $\partial\mathbb{D}_*$ to be 2π which gives the $g_9(\nu)$.

3.3. **The case where $-\infty < \gamma < 0$.** In this case there is no positive separatrix and there are four qualitatively different solutions as pictured in Figure 3.3.

3.3.1. $\mu < 0$. In this case the separatrix $a \equiv 0$ is unstable and so solutions starting above it are strictly increasing, moreover there exists a solution a_4^- which satisfies $a_4^-(0) = 1$. Qualitatively this solution is the same as the solution a_4^+ which satisfies $a_4^+(0) = 1$. Indeed, by the β symmetry, we may take the blow-up time to be $T_1 = \frac{1}{4}$ and so $\mu = \mu(\gamma)$ and $\lambda = \lambda(\gamma)$. Hence, the metrics $g_4^-(\gamma)$ then have similar properties as the metrics produced in Section (3.2.1). However, as $\gamma \to -\infty$ the metrics $g_4^-(\gamma)$ converge to the round metric on the hemi-sphere and so $\nu = \mathrm{dist}(0, \partial\mathbb{D}) \in (\frac{\pi}{2}, \infty)$.

3.3.2. $\mu > 0$. In this case the separatrix $a \equiv 0$ is stable and so solutions starting above it are decreasing and $a(t) \to 0$ as $t \to \infty$. An ODE comparison argument implies that solutions blow-up at some finite initial time T_0. The geometry of the associated metrics is quite different depending on whether $T_0 < 0$, $T_0 = 0$ or $T_0 > 0$.

If $T_0 < 0$, then there is a solution a_{10} which satisfies $a_{10}(0) = 1$ and so, by Lemma 5 and Lemma 6, there exists a family of negatively curved, incomplete metrics $g_{10}(\nu)$ with unbounded curvature on $\mathbb{R}^2$ which are asymptotic to $g_2(\nu)$.

If $T_0 = 0$, then any solution a_{11} has the following asymptotics near $T_0 = 0$

$$a_{11}(t) = \frac{1}{\sqrt{-4\lambda t}} + o\left(\frac{1}{\sqrt{t}}\right)$$

and so $\frac{a_{11}(t)}{t}$ is not locally integrable. By the β symmetry we may take $\lambda = -1$. Hence, by Lemma 6, there is a family of negatively curved, incomplete metrics with unbounded curvature on a cylinder. These metrics are asymptotic at one end to a hyperbolic cusp and asymptotic at the other end to a member of the family $g_2(\nu)$ and give the family $g_{11}(\nu)$ as claimed.

If $T_0 > 0$, then any solution a_{12} has the following asymptotics near $T_0 > 0$

$$a_{12}(t) = \frac{1}{\sqrt{-4\lambda(t - T_0)}} + o\left(\frac{1}{\sqrt{t - T_0}}\right)$$

and so $\frac{a_{12}(t)}{t}$ is locally integrable. Hence, by Lemma 6, there is a family of negatively curved, incomplete metrics with unbounded curvature on $\mathbb{D}^*$. As with g_4, these metrics smoothly extend to $\bar{\mathbb{D}}_*$ and $\partial\bar{\mathbb{D}}_*$ are totally geodesic. By the β-action $\partial_1\mathbb{D}_*$ may be taken to have length 2π. At the other end, these metrics are asymptotic to a member of the $g_2(\nu)$ family at the puncture and give the family $g_{12}(\nu)$ as claimed.

This completes the analysis needed to prove Theorem 1.

3.4. **Proof of Theorem 2.** We now prove Theorem 2. First a simple lemma:

Lemma 7. *Suppose $b \in C^\infty((0, R))$ is given by Lemma 5. If $g = \mathrm{d}r^2 + b(r)^2\mathrm{d}\theta^2$ is a non-constant curvature metric on $\mathbb{D}_*(R)$, then $Isom^+(\mathbb{D}_*(R), g)$, the group of orientation-preserving isometries, is isomorphic to $\mathbb{S}^1$.*

Proof. Clearly, for each $\phi \in \mathbb{S}^1$, the maps $\psi_\phi : (r, \theta) \mapsto (r, \theta + \phi)$ are orientation preserving isometries. Hence, $\mathbb{S}^1$ may be identified with a subgroup of $Isom^+(\mathbb{D}_*(R), g)$. On the other hand, as solutions to (2.7) are either strictly monotone or constant curvature, (2.10) implies that if $\psi : (\mathbb{D}_*(R), g) \to (\mathbb{D}_*(R), g)$ is an isometry, then $r(p) = r(\psi(p))$ and so $\psi = \psi_\phi$ for some ϕ. □

Proof of Theorem 2. As M_* is non-empty, Proposition 4 implies that $K \neq 0$ and $M \backslash M_*$ consists of isolated points $p_i \in M$ – in particular, M_* is connected.

By the analysis of Section 2 and Theorem 1, for each point $p \in M_*$ there is a simply-connected neighborhood $V_p \subset M_*$ about p and an isometric embedding $\imath_p : V_p \to N_p$ where (N_p, g_p) is a scaling of a member of one of the twelve families of by Theorem 1. Clearly, for $q \in M_*$ with $V_p \cap V_q \neq \emptyset$, we have $(N_p, g_q) = (N_q, g_q)$ and there is an element $\psi_{p,q} \in Isom(N_p, g_p)$ so that $\psi_{p,q} \circ \imath_q = \imath_p$ on $V_p \cap V_q$. As M_* is connected, we conclude that $(N_p, g_p) = (N, \beta^2 g)$ for $\beta > 0$ and (N, g) a fixed member of one of the families of Theorem 1. As $p \in M_*$, we may delete a point – if needed – and take $(N, g) = \mathbb{D}_*(R)$ for $0 < R \leq \infty$ and $g = \mathrm{d}r^2 + b(r)^2 \mathrm{d}\theta^2$.

Given points $p, q \in M_*$, a continuous path $\gamma : [0, 1] \to M_*$ connecting p and q, and a fixed isometric embedding $\imath_p : V_p \to M$ there is a unique isometric embedding $\imath_\gamma : V_q \to N$ determined by $\imath_p$ and γ in the obvious way – this is a sort of developing map into (N, g). For a fixed point $p_0 \in M_*$, there is a group homomorphism $\Gamma : \pi_1(M_*, p_0) \to Isom(N, g)$ given by the holonomy of the above construction which is the obstruction to extending a choice of $\imath_{p_0} : V_{p_0} \to N$ to an isometric immersion $\imath : M_* \to N$. As M is orientable as are (by inspection) all the models of Theorem 1 we have that $\Gamma : \pi_1(M_*, p_0) \to Isom^+(N, g)$. Furthermore, as $\pi_1(M_*)$ is cyclic, there is a generator $x \in \pi_1(M_*, p_0)$. We consider $\psi = \Gamma(x)$. By Lemma, 7 there is a $0 < \phi \leq 2\pi$ so $\psi = \psi_\phi$ where $\psi_\phi : (r, \theta) \mapsto (r, \theta + \phi)$. Setting $\alpha = 2\pi\phi^{-1}$ concludes the proof. $\square$

4. A VARIATIONAL CHARACTERIZATION OF TWO-DIMENSIONAL GRADIENT RICCI SOLITONS

Let M denote a smooth two-manifold. On the space of Riemannian metrics $\mathcal{M}_*(M)$ of non-vanishing Gauss curvature on M consider the functional defined by

$$\mathcal{E}[g] = \int_M K_g \log |K_g| \mu_g$$

where μ_g is the area-form of g. This functional has been applied to the study of Ricci flow on surfaces by Hamilton [9] and Chow [5] – in particular Hamilton observed that it is monotonically increasing along the Ricci flow on spheres with positive Gauss curvature.

In [2], we observed that if g is a positively curved metric which is critical for $\mathcal{E}$ with respect to compactly supported conformal variations, then $\hat{g} = K_g^{-3/2} g$ is the metric of a minimal surface in Euclidean 3-space and that every negatively curved minimal surface metric arises in this way. As such, it is interesting to consider critical points of $\mathcal{E}$ under other variations. In this direction, we show that gradient Ricci soliton metrics are the only critical points of $\mathcal{E}$ with respect to area-preserving variations.

We say that $g \in \mathcal{M}_*(M)$ is an $\mathcal{E}$-critical metric if $\mathcal{E}$ is stationary at g with respect to compactly supported area-preserving variations. Let g be a smooth Riemannian metric and h a smooth symmetric 2-form on M. Writing $g_t = g + th$, we have

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \mu_{g_t} = \frac{1}{2} H \, \mu_g.$$

where $H = \mathrm{tr}_g h$ is the trace of h with respect to g. Hence, the variations that preserve area are precisely the deformations by trace-free symmetric 2-forms. Moreover, we have (cf. [7, p. 99])

$$\frac{\partial}{\partial t}\bigg|_{t=0} K_{g_t} = -\frac{1}{4}\Delta_g H + \frac{1}{2}\,\mathrm{div}_g\left(\mathrm{div}_g \mathring{h}\right) - \frac{1}{2} H K_g$$

where $\mathring{h}$ denotes the trace-free part of h. A short computation yields

$$\delta_h \mathcal{E}[g] = \frac{1}{2}\int_M H K_g \log|K_g| + \left(\mathrm{div}_g\left(\mathrm{div}_g \mathring{h}\right) - \frac{1}{2}\Delta_g H - H K_g\right)(\log|K_g|+1)\mu_g.$$

If h is compactly supported then integrating by parts gives

$$\delta_h \mathcal{E}[g] = -\frac{1}{4}\int_M H\left(\Delta_g \log|K_g| + 2K_g\right) - 2\langle h, \mathring{\nabla}^2 \log|K_g|\rangle_g \mu_g, \tag{4.1}$$

where $\langle a, b\rangle_g$ denotes the natural bilinear pairing on elements $a, b \in \Gamma(S^2(T^*M))$ obtained via g. For h trace-free (4.1) simplifies to

$$\delta_h \mathcal{E}[g] = \frac{1}{2}\int_M \langle h, \mathring{\nabla}^2 \log|K_g|\rangle_g\, \mu_g.$$

Hence, g is an $\mathcal{E}$-critical metric if and only if the Gauss curvature K of g satisfies

$$\mathring{\nabla}^2 \log|K| = 0.$$

The functional $\mathcal{E}$ is diffeomorphism invariant, i.e. for $\phi : M \to M$, a diffeomorphism,

$$\mathcal{E}[\phi^* g] = \mathcal{E}[g]$$

for every $g \in \mathcal{M}_*(M)$. Using Noether's principle we obtain a conservation law for $\mathcal{E}$: Let $X \in \Gamma(TM)$ be a compactly supported vector field and let ϕ_t denote its time t flow. Then we have

$$\phi_t^* g = g + t L_X g + o(t).$$

As a consequence of the diffeomorphism invariance we obtain at an $\mathcal{E}$-critical metric

$$\begin{aligned} 0 = \delta_{L_X g}\mathcal{E}[g] &= -\frac{1}{4}\int_M \mathrm{tr}_g(L_X g)\left(\Delta_g \log|K_g| + 2K_g\right)\mu_g, \\ &= -\frac{1}{2}\int_M \left(\Delta_g \log|K_g| + 2K_g\right)\mathrm{div}_g X\, \mu_g \end{aligned}$$

which vanishes for every compactly supported vector field X on M if and only if

$$\Delta_g \log|K_g| + 2K_g$$

is constant. Using Proposition 3 we have thus shown:

Theorem 3. *Let (M, g) be a Riemannian two-manifold with non-zero Gauss curvature. Then (M, g) is a gradient Ricci soliton if and only if g is a critical point of the functional $\mathcal{E}$ with respect to compactly supported area-preserving variations.*

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